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Analytical Description of Stochastic Field-Line Wandering in Magnetic Turbulence

A. Shalchi & I. Kourakis

Institut für Theoretische Physik, Lehrstuhl IV: Weltraum- und Astrophysik,

Ruhr-Universität Bochum, D-44780 Bochum, Germany

(Dated: March 15, 2007)

Abstract

A nonperturbative nonlinear statistical approach is presented to describe turbulent magnetic systems embedded in a uniform mean magnetic field. A general formula in the form of an ordinary differential equation for magnetic field-line wandering (random walk) is derived. By considering the solution of this equation for different limits several new results are obtained. As an example, it is demonstrated that the stochastic wandering of magnetic field-lines in a two-component turbulence model leads to superdiffusive transport, contrary to an existing diffusive picture. The validity of quasilinear theory for field-line wandering is discussed, with respect to different turbulence geometry models, and previous diffusive results are shown to be deduced in appropriate limits.

PACS numbers: 47.27.tb, 96.50.Ci, 96.50.Bh

I. INTRODUCTION

Understanding turbulence occupies a central part of current research efforts in space physics and astrophysics [1, 2]. In an effort to elucidate transport mechanisms in turbulent (collisionless) magnetized plasmas, some progress has been marked by employing a statistical description of turbulence (e.g. [3, 4, 5]). The random component of erratic charged particle motion has recently been associated to the stochastic wandering (random walk) of magnetic field-lines [6, 7]. It is nevertheless understood that none of these efforts has provided a definite, widely applicable analytical formulation of field-line wandering.

A quasi-linear approach for field-line random-walk formed the model basis in early works [8]. In that description, the unperturbed field-lines are used to describe field-line wandering by using a perturbation method. This approach is believed to be correct in the limit of weak turbulence, where turbulent fields are assumed to be much weaker than the (uniform) mean field ($\delta B_i \ll B_0$). A non-perturbative statistical description of field-line wandering was later suggested [9], relying on certain assumptions about the properties of the field-lines (e.g. Gaussian statistics) in combination with an explicit diffusive hypothesis for the field-line topology.

The main scope of this article is to address the problem of field-line wandering analytically, in relation with general turbulence models. The limits of the validity of quasilinear theory (QLT) are also to be discussed. The standard statistical description of field-line wandering is rigorously shown to lead to an ordinary differential equation (ODE) for the mean square deviation (MSD) of the field-lines. The ODE is solved in certain cases and the results are compared with QLT and associated methods. An interesting example is field-line random walk in two-component turbulence, where a superdiffusive behavior of field-line wandering is clearly found.

II. AN ODE FOR FIELD-LINE WANDERING

We shall consider a collisionless magnetized plasma system which is embedded in a uniform mean field ($\vec{B}_0 = B_0\vec{e}_z$) in addition to a turbulent magnetic field component in the transverse direction ($\delta B_z = 0$). The field-line equation in this system reads $dx/dz = \delta B_x/B_0$. Following the established Kubo statistical formalism for random processes, the field-line (FL) mean square displacement (MSD) can be written as

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{1}{B_0^2} \int_0^z dz' \int_0^z dz'' \, R_{xx}(z', z'') \tag{1}$$

where we have employed $\Delta x = x(z) - x(0)$ and the $xx-$component of the magnetic correlation tensor $R_{xx}(z', z'') = \langle \delta B_x(\vec{x}(z'))\delta B_x^*(\vec{x}(z''))\rangle$; the real part of the right-hand side (*rhs*) will be understood. Assuming homogeneous turbulence $R_{xx}(z', z'') = R_{xx}(|z' - z''|)$, one obtains

$$\left\langle (\Delta x(z))^2 \right\rangle = \frac{2}{B_0^2} \int_0^z dz' \, (z - z') R_{xx}(z') \tag{2}$$

with $R_{xx}(z') = \langle \delta B_x(\vec{x}(z'))\delta B_x^*(\vec{x}(0))\rangle$. Since the correlation tensor is itself dependent of $\vec{x}(z) = (x(z), y(z), z)$ this is an implicitly nonlinear integral equation for the magnetic field-line space topology. In order to evaluate Eq. (2), one has to determine the correlation tensor R_{xx}. A Fourier transformation leads to

$$R_{xx}(z) = \int d^3k \int d^3k' \left\langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}')e^{i\vec{k}\cdot\vec{x}(z) - i\vec{k}'\cdot\vec{x}(0)} \right\rangle , \tag{3}$$

or, adopting Corrsin's independence hypothesis [10]

$$\begin{aligned} R_{xx}(z) \; = & \int d^3k \int d^3k' \left\langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}') \right\rangle \\ & \times \left\langle e^{i\vec{k}\cdot\vec{x}(z) - i\vec{k}'\cdot\vec{x}(0)} \right\rangle . \end{aligned} \tag{4}$$

Assuming that the magnetic fields for different wave vectors are decorrelated $\left\langle \delta B_x(\vec{k})\delta B_x^*(\vec{k}') \right\rangle = P_{xx}(\vec{k})\delta(\vec{k} - \vec{k}')$ one is led to

$$R_{xx}(z) = \int d^3k \, P_{xx}(\vec{k}) \left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle \tag{5}$$

with $P_{xx}(\vec{k}) =< \delta B_x(\vec{k})\delta B_x^*(\vec{k}) >$. For the sake of analytical tractability, in order to evaluate the characteristic function $< e^{i\vec{k}\cdot\Delta\vec{x}(z)} >$, we shall assume Gaussian statistics for the field-lines, thus

$$\left\langle e^{i\vec{k}\cdot\Delta\vec{x}(z)} \right\rangle = e^{-\frac{1}{2}\left\langle (\Delta x(z))^2 \right\rangle k_x^2 - \frac{1}{2}\left\langle (\Delta y(z))^2 \right\rangle k_y^2 + ik_\parallel z}. \tag{6}$$

For axisymmetric turbulence $\langle (\Delta x)^2 \rangle = \langle (\Delta y)^2 \rangle$, so Eq. (2) takes the form

$$\begin{aligned} \left\langle (\Delta x(z))^2 \right\rangle = & \frac{2}{B_0^2} \int d^3k \, P_{xx}(\vec{k}) \\ \times & \int_0^z dz' \, (z - z') \cos(k_\parallel z') e^{-\frac{1}{2}\left\langle (\Delta x(z'))^2 \right\rangle k_\perp^2}. \end{aligned} \tag{7}$$

Upon differentiation with respect to z, we find for the field-line MSD

$$\frac{d}{dz}\left\langle(\Delta x(z))^2\right\rangle = \frac{2}{B_0^2}\int d^3k\ P_{xx}(\vec{k}) \times \int_0^z dz'\ \cos(k_\parallel z')e^{-\frac{1}{2}\left\langle(\Delta x(z'))^2\right\rangle k_\perp^2}. \tag{8}$$

A second differentiation leads to

$$\frac{d^2}{dz^2}\left\langle(\Delta x(z))^2\right\rangle = \frac{2}{B_0^2}\int d^3k\ P_{xx}(\vec{k}) \times \cos(k_\parallel z)e^{-\frac{1}{2}\left\langle(\Delta x(z))^2\right\rangle k_\perp^2}. \tag{9}$$

This ODE was obtained exactly, relying on no other assumptions than Corrsin's hypothesis and Gaussian FL statistics. In combination with different turbulence models, it provides a general basis for the determination of the FL-MSD, thus allowing for a quantitative description of field-line wandering. In the following, we shall consider a specific example, as well as various limits of this description.

III. TWO-COMPONENT TURBULENCE DESCRIPTION

A two-component turbulence model has been proposed as a realistic model for solar wind turbulence [11]. Within this model, the turbulent fields are described as a superposition of a slab model ($\vec{k} \parallel \vec{B}_0$) and a two-dimensional (2D) model ($\vec{k} \perp \vec{B}_0$). In the following, we shall evaluate the field-line MSD, in view of identifying an comparing among pure slab, pure 2D, and composite slab/2D geometry.

A. Analytical results

Assuming *2D* turbulence statistics, the $xx-$component of the correlation tensor reads

$$P_{xx}^{2D}(\vec{k}) = g^{2D}(k_\perp)\frac{\delta(k_\parallel)}{k_\perp}\left(1-\frac{k_x^2}{k^2}\right). \tag{10}$$

Eq. (9) thus becomes

$$\frac{d^2}{dz^2}\left\langle(\Delta x)^2\right\rangle = \frac{2\pi}{B_0^2}\int_0^\infty dk_\perp\ g^{2D}(k_\perp)e^{-\frac{1}{2}\left\langle(\Delta x)^2\right\rangle k_\perp^2}. \tag{11}$$

Focusing on high values of the position variable $z \to \infty$, hence $\langle(\Delta x)^2\rangle \to \infty$, the main contribution to the integral comes from low values of the integration variable. Assuming a quasi-constant behavior of the spectrum in the energy-range we may approximate as

$$\begin{aligned}\frac{d^2}{dz^2}\left\langle (\Delta x)^2 \right\rangle &\simeq \frac{2\pi}{B_0^2} g^{2D}(0) \int_0^\infty dk_\perp \, e^{-\frac{1}{2}\left\langle (\Delta x)^2 \right\rangle k_\perp^2} \\ &= \frac{2\pi}{B_0^2}\sqrt{\frac{\pi}{2}} g^{2D}(0) \left[\left\langle (\Delta x)^2 \right\rangle\right]^{-1/2} .\end{aligned} \tag{12}$$

A more tedious yet exact analysis of the integral in (11) also leads to the same qualitative result. It is straightforward to show that Eq. (12) is solved by

$$\left\langle (\Delta x)^2 \right\rangle \simeq \left[\frac{9\pi}{2B_0^2}\sqrt{\frac{\pi}{2}} g^{2D}(0)\right]^{2/3} z^{4/3} \tag{13}$$

in the limit $z \to \infty$.

In recent turbulence and cosmic ray studies, the form

$$g^{2D}(k_\perp) = \frac{2C(\nu)}{\pi} l_{2D} \delta B_{2D}^2 \left(1 + k_\perp^2 l_{2D}^2\right)^{-\nu} \tag{14}$$

has been proposed [11]. Here $C(\nu)$ is a normalization constant, l_{2D} is the 2D-bendover-scale, 2ν is the inertial-range spectral index, and $\delta B_{2D}^2/B_0^2$ determines the relative strength of 2D turbulence. Combining this model spectrum with the above, we find

$$\left\langle (\Delta x)^2 \right\rangle = \left[9C(\nu)\sqrt{\frac{\pi}{2}} l_{2D} \frac{\delta B_{2D}^2}{B_0^2}\right]^{2/3} z^{4/3} \tag{15}$$

which is clearly a *non-diffusive* result.

A different approach consists in assuming *slab* turbulence statistics. The $xx-$component of the correlation tensor reads

$$P_{xx}^{slab}(\vec{k}) = g^{slab}(k_\parallel) \frac{\delta(k_\perp)}{k_\perp} , \tag{16}$$

where we assume

$$g^{slab}(k_\parallel) = \frac{C(\nu)}{2\pi} l_{slab} \delta B_{slab}^2 (1 + k_\parallel^2 l_{slab}^2)^{-\nu} . \tag{17}$$

We have defined the slab-bendover-scale l_{slab}, and the relative strength of slab turbulence $\delta B_{slab}^2/B_0^2$. For the spectrum of Eq. (17), it is straightforward to show that the slab result behaves diffusively, as

$$\left\langle (\Delta x(z))^2 \right\rangle = 2\kappa_{slab}\, z \tag{18}$$

with

$$\kappa_{slab} = \pi C(\nu) l_{slab} \frac{\delta B_{slab}^2}{B_0^2} . \tag{19}$$

This is an exact result, readily obtained upon analytical evaluation of Eq. (7) in the limit $z \to \infty$.

In a *hybrid (composite) slab/2D* model, the correlation tensor is assumed to have the form $P_{xx}(\vec{k}) = P_{xx}^{slab}(\vec{k}) + P_{xx}^{2D}(\vec{k})$ (following the definitions above). It can easily be shown (and was indeed confirmed numerically) that the diffusive slab contribution, within the composite turbulence model, is negligible in comparison to the superdiffusive 2D contribution. Thus, Eq. (15) may solely be employed for the description of field-line wandering, also in two-component turbulence.

B. Numerical evaluation

Eq. (7) is an integral-equation for the MSD of the field-lines. By using Eqs. (10) and (14) for $P_{xx}^{2D}(\vec{k})$, in addition to Eqs. (16) and (17) for $P_{xx}^{slab}(\vec{k})$, the total correlation tensor $P_{xx}(\vec{k})$ for the composite model is specified.

We have evaluated Eq. (7) numerically, adopting a 20% slab-/80% 2D- composite turbulence model. For the sake of comparison, we have also computed the result in the pure-slab turbulence case for $\delta B_{slab}^2/B_0^2 = 1$.

For the same cases of study, we have seen above that the analytical result for two-component turbulence is given by Eq. (15), whereas the pure slab result is given by (18).

The numerical results are compared with the analytical results (presented above) in Fig. 1, by depicting the running diffusion coefficients $\left\langle (\Delta x(z))^2 \right\rangle / (2 z l_{slab})$ as a function of z/l_{slab}. For composite geometry, the analytical result (solid curve) is compared to the numerical result (dashed curve). The analytical slab result (dash-dotted curve) and the numerical slab result (dotted curve) are also provided, for reference. An excellent agreement is witnessed among the theoretical and the numerical results, in both cases. The analytical expression (15) derived above for the asymptotic behavior of the field-line MSD thus appears to be valid in the composite turbulence model also and, in fact, bears a contributions which soon exceeds the diffusive result of the slab-model significantly: cf. the upper curve(s) in Fig. 1 to the lower, constant curve(s).

IV. FURTHER RESULTS AND LIMITS

We shall complete our study by considering various limiting cases, in which previous theoretical results are recovered. Although these results are not new, it is interesting *per se* to demonstrate that they can be obtained as appropriate limits from our theory.

A. The initial free streaming regime

For small values of the position variable ($z \to 0$) hence expecting $\langle(\Delta x)^2\rangle \to 0$, we obtain from Eq. (9)

$$\frac{d^2}{dz^2}\left\langle(\Delta x)^2\right\rangle \approx \frac{2}{B_0^2}\int d^3k \; P_{xx}(\vec{k}) = 2\frac{\delta B_x^2}{B_0^2}. \tag{20}$$

Assuming vanishing conditions for both the MSD and its derivative at zero, one obtains

$$\left\langle(\Delta x)^2\right\rangle = \frac{\delta B_x^2}{B_0^2} z^2 \tag{21}$$

so a strong superdiffusive FL wandering regime is clearly found for small z, independent of the turbulence model adopted.

B. Quasilinear theory for field-line random walk

Within QLT, one may replace the MSD on the right hand side of Eq. (9) by the unperturbed field-lines (thus taking $\langle(\Delta x)^2\rangle = 0$), which yields

$$\frac{d^2}{dz^2}\left\langle(\Delta x)^2\right\rangle = \frac{2}{B_0^2}\int d^3k \; P_{xx}(\vec{k})\cos(k_\parallel z). \tag{22}$$

QLT is thus apparently only exact for pure-slab turbulence, where $P_{lm} \sim \delta(k_\perp)$; cf. Eq. (16). On the other hand, within QLT one finds for pure-2D turbulence [adopting Eqs. (10) and (22)] that $\langle(\Delta x)^2\rangle = (\delta B_x/B_0)^2 z^2$, in disagreement with the nonlinear result obtained above. Thus, QLT fails to describe field-line wandering in the two-component model. Presumably, quasilinear theory might thus also not apply in other non-slab models.

C. The diffusion limit

Combining Eq. (8) with the assumption of diffusion for the magnetic field-lines (i.e. explicitly setting $\langle (\Delta x)^2 \rangle = 2\kappa z$) one obtains exactly

$$\kappa = \frac{1}{B_0^2} \int d^3k \; P_{xx}(\vec{k}) \int_0^\infty dz \; \cos(k_\parallel z) e^{-\kappa z k_\perp^2}. \tag{23}$$

The $z-$integral can easily be solved to give

$$\kappa = \frac{1}{B_0^2} \int d^3k \; P_{xx}(\vec{k}) \frac{\kappa k_\perp^2}{k_\parallel^2 + (\kappa k_\perp^2)^2}. \tag{24}$$

This formula is correct if field-line random walk behaves diffusively and if the small length scales of the initial free streaming regime are unimportant.

For two-component turbulence Eq. (24) can easily be evaluated; we find

$$\kappa = \kappa_{slab} + \frac{\kappa_{2D}^2}{\kappa}\,, \tag{25}$$

using the slab diffusion coefficient κ_{slab} of Eq. (19) and

$$\kappa_{2D}^2 = \frac{1}{B_0^2} \int d^3k \; k_\perp^{-2} P_{xx}^{2D}(\vec{k})\,. \tag{26}$$

Eq. (25) is a quadratic equation in κ, which may easily be solved to get

$$\kappa = \frac{\kappa_{slab} + \sqrt{\kappa_{slab}^2 + 4\kappa_{2D}^2}}{2}\,. \tag{27}$$

We note that this coincides with the result derived earlier by Matthaeus *et al.* [9]. For pure slab turbulence, we have $\kappa_{2D} = 0$, so the expected limit $\kappa = \kappa_{slab}$ is recovered. On the other hand, for pure 2D geometry, i.e. for $\kappa_{slab} = 0$, one finds $\kappa = \kappa_{2D}$. Thus the parameter κ_{2D} can be identified with the *"diffusion coefficient"* for pure 2D turbulence. However, adopting the standard spectrum of Eq. (14), one clearly obtains a diverging result, as $\kappa_{2D} \to \infty$. In the light of the results presented above, the reason for this divergent behavior may be the *superdiffusive* nature of field-line random walk. It appears that Eq. (27) fails to provide an appropriate description of the composite turbulence, in contrast with the generalized nonlinear theory presented in this paper.

V. CONCLUSION

A generalized nonlinear formulation has been presented, to describe field-line wandering (random walk) in general magnetostatic systems consisting of a statistical (or turbulent) component δB and a nonstatistical uniform component B_0. Assuming vanishing parallel component of the turbulent field ($\delta B_z = 0$), applying the Corrsin approximation, and assuming Gaussian statistics, a general ODE was deduced for the field-line MSD (Eq. (9)) in axisymmetric and homogeneous turbulent plasmas.

Adopting the two-component turbulence model which was suggested by Bieber et al. ([11]) as a realistic model for solar wind turbulence. It was demonstrated systematically that field-line wandering behaves superdiffusively. Specifically, a weakly *superdiffusive* behavior of the mean square deviation was found in the form $< (\Delta x)^2 >\sim z^{4/3}$. This result was compared to previous results obtained via different assumptions, namely the quasilinear result ($< (\Delta x)^2 >\sim z^2$) and the diffusive result $< (\Delta x)^2 >\sim z$ of Ref. [9].

Extending the considerations in this article, the generalized nonlinear formulation presented here may be applied in a description of non-axisymmetric turbulence and/or other (e.g., anisotropic) turbulence geometries.

Since field-line random walk is of major importance in the description transport of charged particles in astrophysical plasmas, these results are significant in cosmic ray space physics and astrophysics. Concluding, the analytical expression(s) derived in this article provide a useful toolbox, which extends existing transport theoretical models and might be essential in inspiring forthcoming ones.

Acknowledgments

This research was supported by Deutsche Forschungsgemeinschaft (DFG) under the Emmy-Noether Programme (grant SH 93/3-1). As a member of the *Junges Kolleg* A. Shalchi also aknowledges support by the Nordrhein-Westfälische Akademie der Wissenschaften.

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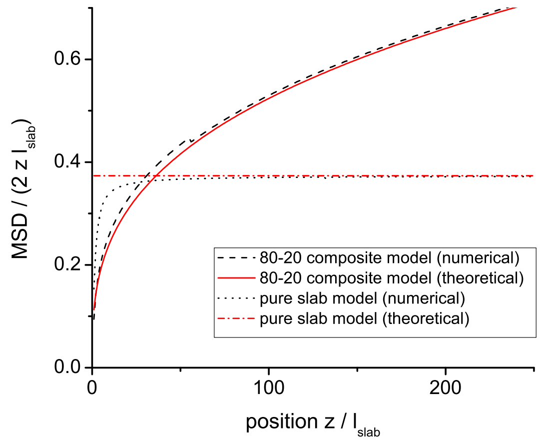

FIG. 1: The running diffusion coefficients $\left\langle (\Delta x(z))^2 \right\rangle / (2zl_{slab})$ as a function of z/l_{slab} are depicted. The analytical result (solid line) is compared to the numerical result (dashed line) for 20 % slab / 80 % 2D composite geometry. The analytical slab result (dash-dotted line) and the numerical slab result (dotted line) are also provided, for reference. The analytical and numerical results are are obviously in good agreement.